Chapter 3

Summary and Vocabulary

- Equations, graphs of functions, and data can be transformed in similar ways. Two such **transformations** – **translations** and **scale changes** – are studied in this chapter. When graphs of functions are translated or scaled, the **images** resemble the graphs of the **preimages**, or original **parent functions**. Translations slide graphs, whereas scale changes stretch or shrink them horizontally and vertically.
- Connections among transformations, function equations, and graphs are given by the Graph-Translation and Graph Scale-Change Theorems. By the Graph-Translation Theorem, a translation $(x, y) \rightarrow (x + h, y + k)$ transforms the graph of $y = f(x)$ to a graph of $y - k = f(x - h)$. The Graph Scale-Change Theorem asserts that scaling the graph of $y = f(x)$ by **horizontal scale factor** a and **vertical scale factor** b, $(x, y) \rightarrow (ax, by)$, produces the graph of $\frac{y}{b} = f\left(\frac{x}{a}\right)$. Some features of graphs can be predicted if just the equation is given, and equations can be written if just the graph is given.
- Symmetries of graphs can be determined from their equations. **Odd functions** have **point symmetry** around the origin; **even functions** are **reflection-symmetric** over the y-axis.
- Asymptotes, discontinuities, and other features of functions can be identified from a graph of the function. Knowing the parent function of a transformed function can assist in this identification.
- Under a translation of magnitude h, measures of center for a data set are translated by h, while measures of spread are unaffected. In contrast, when data are multiplied by a factor $a > 0$, measures of center, the standard deviation, and the range are multiplied by a, and the variance is multiplied by a^2.
- **Composites** of functions are formed by letting one function g operate on those outputs of another function f that are in g's domain. The composite of f followed by g is written as $g \circ f$, and is defined by $(g \circ f)(x) = g(f(x))$. In general, **function composition** is not commutative. That is, $f \circ g$ and $g \circ f$ are usually different functions.
- The **inverse** of a function $f: x \rightarrow y$ can be obtained by switching x's and y's in its equation or by switching x- and y-coordinates in a set of ordered pairs. When the resulting relation is a function, it is denoted by f^{-1}. The graphs of a function and its inverse are reflection images of each other with respect to the line $y = x$. Another characteristic property of inverse functions is that the composite of a function f and its inverse f^{-1} is the **identity function** I. That is, for all x in the domain of the composite, $f(f^{-1}(x)) = f^{-1}(f(x)) = I(x) = x$.

Vocabulary

3-1
window
parent function

3-2
transformation
preimage
image
translation

3-3
invariant

3-4
reflection-symmetric
axis of symmetry
line of symmetry
symmetry about a point
point symmetry
center of symmetry
even function
odd function

3-5
horizontal and vertical scale change
horizontal and vertical scale factor
size change

3-6
scale change of a data set
scale factor
scale image

3-7
composite
function composition

- Suppose a data set has mean $\bar{x}$ and standard deviation s. A ***z*-score** is the result of a composite of a specific translation $(T(x) = x - \bar{x})$ followed by a scale change $(S(x) = \frac{x}{s})$ of the translated data. The z-score, $z = \frac{x - \bar{x}}{s}$, corresponding to the raw score x, tells how many standard deviations x is above or below the mean. A data set transformed in this way has mean 0 and standard deviation 1. Using z-scores makes it possible to compare scores from different data sets.

Vocabulary

3-8

inverse of a function
identity function

3-9

z-score
raw data
standardized data

Properties and Theorems

Graph-Translation Theorem (p. 161)
Theorem (Centers of Translated Data) (p. 167)
Theorem (Spreads of Translated Data) (p. 168)
Theorem (Symmetry over the *y*-axis) (p. 173)
Theorem (Symmetry over the *x*-axis) (p. 173)
Theorem (Symmetry about the Origin) (p. 173)
Graph Scale-Change Theorem (p. 181)
Theorem (Centers of Scaled Data) (p. 188)
Theorem (Spreads of Scaled Data) (p. 189)
Inverses of Functions Theorem (p. 202)
Theorem (Mean and Standard Deviation of *z*-scores) (p. 207)

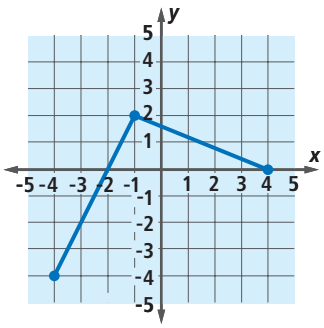
Chapter 3 Self-Test

Take this test as you would take a test in class. You will need a calculator. Then use the Selected Answers section in the back of the book to check your work.

1. Give equations for the asymptotes of the graph of the function $h: x \rightarrow \frac{1}{x-7} - 9$.

In 2 and 3, let the translation T: $(x, y) \rightarrow (x + 4, y - 2)$ be applied to the graph of $y = x^2$.

2. Write an equation for the image.
3. What are the coordinates of the vertex of the image?
4. The graph of a function f is below. Sketch a graph of its image under the transformation S when $S(x, y) = (2x, -y)$.



In 5 and 6, suppose a chemistry student finds masses of different samples of potassium chloride (KCl) and subtracts 150 from each value before computing the statistics below.

mean: 4.2 g	standard deviation: 1.3 g
minimum: 0.4 g	maximum: 13.6 g
median: 3.8 g	Q_1: 3.6 g Q_3: 4.6 g

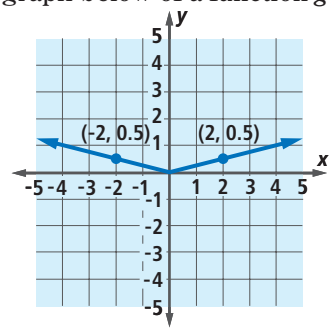
5. For the actual sample masses, give the
   a. median. b. range. c. IQR.
6. For the actual sample masses, give each statistic from Question 5 in ounces using 1 gram $\approx$ 0.035 ounces.

In 7–9, let f be a real function with $f(x) = 4\sqrt[3]{x+2}$.

7. Find an equation for the inverse of f.
8. Is the inverse of f a function? Support your answer.
9. Sketch graphs of f and its inverse.

In 10 and 11, let m and n be real functions with $m(x) = 16 - 5x$ and $n(x) = x + \sqrt{x}$.

10. Write an expression for $n(m(x))$.
11. Give the domain of $n \circ m$.
12. Suppose the scale change $S(x, y) = \left(3x, \frac{y}{2}\right)$ is applied to the graph of $y = \frac{1}{x^2}$. Write an equation for the image.
13. A data set has a mean of 83 and a standard deviation of 7. What transformation should be applied so that the image set has a mean of 0 and a standard deviation of 1?
14. Tell whether the function j with equation $j(x) = x^2 - 5$ is even, odd, or neither. Support your answer with algebra.
15. The graph below of a function g is an angle.



   a. What is the parent function of g?
   b. Find an equation for g.
   c. What symmetries does the graph of g have?
   d. Give the range of g.
16. Give a rule for the reflection over the line $y = x$.
17. Suppose a population of mice has a mean weight of 30 g and a standard deviation of 3.4 g. A population of moose has a mean weight of 910 lb and a standard deviation of 185 lb. Which animal is heavier relative to its population, a 37-g mouse or a 1260-lb moose? Explain your answer.

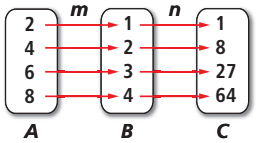
Chapter 3 Chapter Review

SKILLS
PROPERTIES
USES
REPRESENTATIONS

SKILLS Procedures used to get answers

OBJECTIVE A Find equations for and values of composites of functions. (Lesson 3-7)

In 1 and 2, consider the functions m mapping A to B, and n mapping B to C, as shown at the right.



1. What is $n(m(6))$?
2. Find $(n \circ m)(2)$.

In 3 and 4, let $f(t) = 7t - 9$ and $g(t) = t^2 - t$.

3. Evaluate each composite.
   a. $f(g(-2))$ b. $g(f(-2))$
4. Find a formula for $f(f(t))$.

In 5 and 6, let $f(x) = \frac{4}{x}$ and $g(x) = x - 8$.

5. Evaluate each composite.
   a. $(f \circ g)(-8)$ b. $(g \circ g)(-8)$
6. Find an equation for $g \circ f$.

OBJECTIVE B Find inverses of functions. (Lesson 3-8).

In 7–9, a function is described.

a. Describe the inverse using a set of ordered pairs or an equation.

b. State whether the inverse is a function.

7. $f(x) = |x|$
8. $\ell(x) = \frac{4}{x - 3}$
9. $g = \{(3, 4), (4, 12), (5, 9), (6, 0), (7, 11)\}$

PROPERTIES Principles behind the mathematics

OBJECTIVE C Use the Graph-Translation Theorem or the Graph Scale-Change Theorem to find transformation images. (Lessons 3-2, 3-5)

10. **Multiple Choice** Which scale change stretches a graph horizontally by a factor of 4 and shrinks it vertically by a factor of 11?
    A $S(x, y) = (4x, 11y)$ B $S(x, y) = \left(\frac{x}{4}, 11y\right)$
    C $S(x, y) = \left(\frac{x}{4}, \frac{y}{11}\right)$ D $S(x, y) = \left(4x, \frac{y}{11}\right)$
11. **Multiple Choice** Which translation has the effect on a graph of moving each point 5 units down and 9 units to the right?
    A $T(x, y) = (x - 5, y + 9)$
    B $T(x, y) = (x + 9, y - 5)$
    C $T(x, y) = (x - 9, y + 5)$
    D $T(x, y) = (x + 5, y - 9)$

In 12 and 13, find an equation for the image of the graph of $y = x^2$ under the transformation.

12. T: $(x, y) \rightarrow (x - 2, y + 7)$
13. S: $(x, y) \rightarrow \left(\frac{x}{2}, 3y\right)$

In 14 and 15, suppose $f(x) = |x|$. Find an equation for the image of the graph of f under the transformation.

14. $S(x, y) = \left(5x, \frac{y}{4}\right)$
15. $T(x, y) = (x - 1, y)$
16. Describe a transformation that maps the graph of $\sqrt{x}$ onto the graph of $\sqrt{5x}$.
17. Describe a transformation that maps the graph of $y = 8^x$ onto the graph of $y = 8^x + 4$.

OBJECTIVE D Describe the effects of translations and scale changes on functions and their graphs. (Lessons 3-2, 3-4, 3-5)

In 18–21, describe how the graph of the image is related to the graph of the preimage when the given change is made in the equation for a function or relation.

18. x is replaced by $x - 90$.
19. y is replaced by $y + 15.3$.
20. x is replaced by $3x$.

21. y is replaced by $\frac{y}{8}$.

22. Match each general transformation with one of Questions 18–21 above.

a. T: $(x, y) = (x, y + k)$ b. T: $(x, y) \rightarrow (x, by)$

c. T: $(x, y) = (ax, y)$ d. T: $(x, y) \rightarrow (x + h, y)$

23. **True or False** Under a translation, asymptotes of the preimage are mapped to asymptotes of the image.

24. Give a rule for a scale change that has the effect of reflecting the graph over the x-axis.

OBJECTIVE E Describe and identify symmetries and asymptotes of graphs. (Lessons 3-1, 3-4)

In 25–28, a function is described by an equation. Determine if the function is odd, even, or neither.

25. $k(t) = |3t + 4|$ 26. $j(m) = 10m^3$

27. $f(x) = 3|x| + 1$ 28. $s(y) = 7y^2 - 3y^4$

29. Consider the function f with $f(x) = \frac{1}{2x - 1}$. Give equations for any asymptotes of its graph.

30. If a graph has a horizontal asymptote at $y = 3$ and a vertical asymptote at $x = -2$, around what point might you center the screen to see the graph on your calculator?

OBJECTIVE F Identify properties of composites and inverses of functions. (Lessons 3-7, 3-8)

31. If f and g are real functions with $f(x) = \sqrt{x}$ and $g(x) = x - 2$, what is the domain of $f \circ g$?

32. If two relations are inverses of each other, what transformation maps the graph of one onto the graph of the other?

In 33–37, true or false.

33. If f is a function with an inverse f^{-1}, then $f^{-1}(x) = \frac{1}{f(x)}$.

34. If $f(x) = x^2$ and $g(x) = \sqrt{x} + 2$, for all x, $f \circ g(x) = g \circ f(x)$.

35. If f is a function, then $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

36. For all real functions f and g, $f \circ g = g \circ f$.

37. A scale change with magnitude a followed by a translation of h units is the same as a translation of h units followed by a scale change with magnitude a.

OBJECTIVE G Identify properties of z-scores. (Lesson 3-9)

In 38 and 39, a new data set is formed by taking the z-scores from raw scores with mean $\bar{x}$ and standard deviation s.

38. What is the standard deviation of the standardized data set?

39. What is the mean of the standardized data set?

In 40 and 41, a z-score is given. Explain what it tells you about the original data point in terms of the mean and standard deviation of the original data set.

40. $z = 0.04$ 41. $z = -1.3$

USES Applications of mathematics in real-world situations

OBJECTIVE H Use translations, scale changes, or z-scores to describe and analyze data and statistics. (Lessons 3-3, 3-6, 3-9)

In 42 and 43, consider the table below.

Raw Scores	Scaled Scores	Frequency
4	16	1
5	20	3
6	24	2
8	32	7
9	36	2

42. a. Find the mode, mean, and median of the raw data.

b. Identify the transformation used to scale the scores.

c. Find the mode, mean, and median of the scaled scores.

d. What theorem of translated data is shown in Parts b and c?

43. a. What is the range of the raw scores?
    b. What is the range of the scaled scores?
    c. What theorem of translated data is shown in Parts a and b?

44. Use a translation to mentally calculate the average of these bowling scores: 103, 114, 107, 101, 105.

45. For a sample of a certain butterfly species, a scientist found a mean length of 1.82 inches with a range of 2.71 inches and a standard deviation of 0.46 inches. If the data are converted to millimeters (1 inch = 25.4 mm), give the following statistics of the resulting data set.
    a. mean b. range c. variance

46. Ken scored 83 on an *FST* test on which the mean was 77 and the standard deviation was 4.1. His score on an English test was 63. On the English test, the mean was 54 and the standard deviation was 5.3. On which test did he do better compared to his classmates?

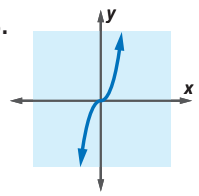
47. A pod of porpoises has a mean weight of 163 kg and a standard deviation of 29 kg. A school of yellow fin tuna has a mean weight of 83 lb and a standard deviation of 17 lb. Which animal is heavier relative to its group, a porpoise which weighs 207 kg or a tuna which weighs 107 lb?

REPRESENTATIONS Pictures, graphs, or objects that illustrate concepts

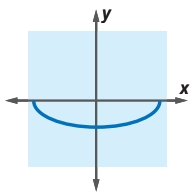
OBJECTIVE I Recognize functions and their properties from their graphs. (Lessons 3-1, 3-4, 3-8)

In 48 and 49, determine whether the function graphed seems to have an inverse which is a function. Give a reason for your answer.

48.



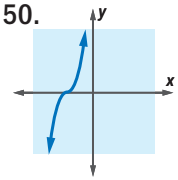
49.



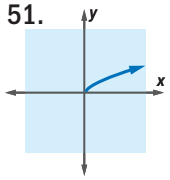
In 50–55, match each graph with the equation of its parent function.

A $y = |x|$ B $y = \sqrt{x}$ C $y = x$
D $y = \frac{1}{x}$ E $y = x^3$ F $y = x^2$

50.



51.



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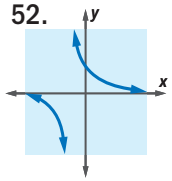

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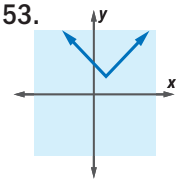

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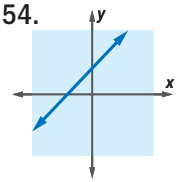
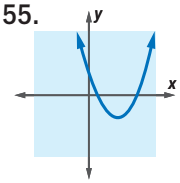
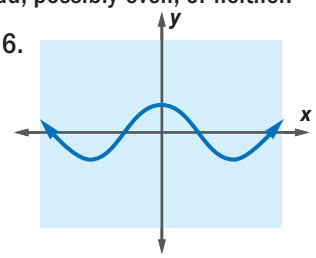


55.



In 56 and 57, classify the function graphed as possibly odd, possibly even, or neither.

56.



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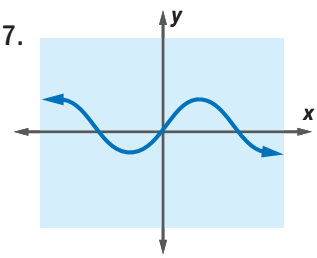
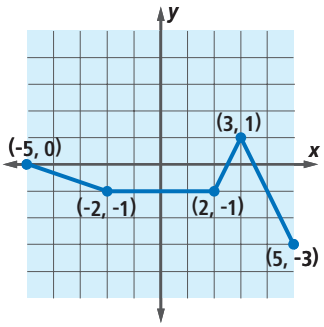


OBJECTIVE J Apply the Graph-Translation Theorem or the Graph Scale-Change Theorem to make or identify graphs. (Lessons 3-2, 3-5)

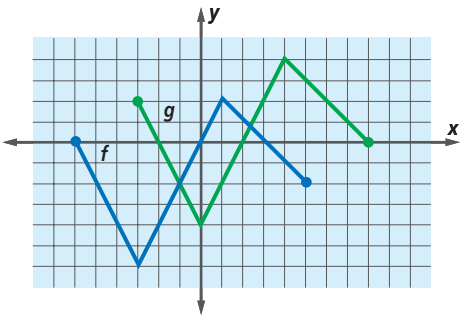
58. Sketch the graph of $y = |2x| + 10$ and its parent function on the same set of axes.

59. Let $k(x) = 12 - \frac{2}{x+4}$.
    a. Sketch a graph of the function k.
    b. Give equations of any asymptotes.
    c. Give the coordinates of the x-intercepts.

60. The graph of $y = f(x)$ is drawn below. Draw the graph of $y = 4f(-x)$.



61. Give an equation for the transformation that maps the graph of f onto the graph of g.

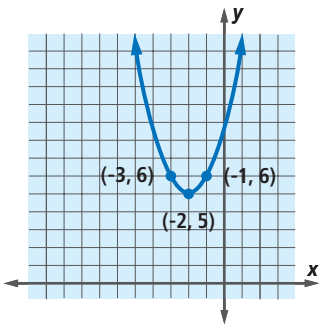


62. a. Graph $f(x) = x^2$ and $g(x) = 2x^2$ on the same set of axes.

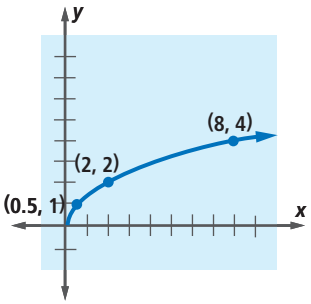
b. True or False g is the image of f under the transformation S: $(x, y) \rightarrow (2x, y)$.

In 63–65, the graph is a translation or a scale-change image of the graph of the given parent function. Write an equation for the function that is graphed.

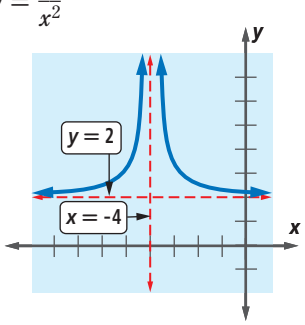
63. parent: $y = x^2$



64. parent: $y = \sqrt{x}$



65. parent: $y = \frac{1}{x^2}$



OBJECTIVE K Graph inverses of functions. (Lesson 3-8)

In 66–68, a rule for a function is given.
a. Find an equation for its inverse.
b. Graph the function and its inverse.
c. Determine if the inverse is a function.

66. $y = -2x^3$ **67.** $h(x) = \frac{2}{x}$ **68.** $j(x) = -x^2 + 4$

69. Tell whether or not f and g graphed below are inverses of each other. Explain your reasoning.